Technical Paper

Visualizing Electromagnetic Vector Fields in Matter using MATHEMATICA

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Abstract

The quantities of dynamic physics including gravitational field and electromagnetic field are accurately described using vector differential operators. Electromagnetism is notably more conceptual in matter compared to vacuum because abstract, three-dimensional fields that are sometimes difficult to visualize are involved in most analysis. A visual representation of abstract vector fields in matter is invaluable to students or researchers working in this field and may also assist teachers in teaching electromagnetism to physics or engineering students. We successfully visualized the most fundamental concepts of the electromagnetic vector calculated and estimated using vector differential operators in *Mathematica*. This visualization based on vector calculations can be used as a starting platform for the further exploration of electromagnetic vector fields.

Keywords: Electromagnetic vector field, Mathematica, Vector differential operators, Mathematica simulation, Vector field visualization

1. Introduction

Electromagnetic vector fields are composed of elaborate mathematical structures and abstract concepts described as vector differential operators [1-5]. It is often conceptually challenging for physicists to intuitively understand the associated physics phenomena and distinguish between them. To learn about the changes in the electric displacement $\vec{D}$ in a medium, including polarization $\vec{P}$ and electric quadrupole $Q_{\alpha, \beta}$, or magnetic intensity $\vec{H}$ in the medium including the magnetization vector $\vec{M}$ in the field $\vec{B}_0$ [5,6], the vector fields must be calculated and visualized to understand their characteristic features. According to our educational experience, the physicists must introduce a visualizing portfolio in curriculum to examine and test these concepts of physics. Several tools are available for the visualization of electromagnetic field vector such as: MATLAB's interactive tools [7,8], Maple's Classroom Tips [9], and *Mathematica* demonstrations [10-14]; that are effectively used to plot vector fields [7-9,12]. However, they are limited to certain fields in vacuum

[9,13] or special materials [15-18], and do not specifying the constitutive relations in matter. The electric and the magnetic fields $\vec{E}$ and $\vec{B}$ respectively, are determined every where in space if, all sources, charge ρ and current density $\vec{J}$, are specified. For a small number of definite sources, the determination of the fields is a tractable problem. However, for most cases of macroscopic aggregates of matter, the $\vec{D}$ and $\vec{H}$ are determined by the macroscopic sources $\vec{P}$, $\vec{H}$, $Q_{\alpha, \beta}$ and similar moment densities of the material medium in the presence of an applied field. The derived fields $\vec{D}$ and $\vec{H}$ are expressed in terms of $\vec{E}$ and $\vec{B}$ using the constitutive relations in the material medium [5,18]. Maxwell equations do not preclude the possibility that one or both of quantities ϵ , μ , be negative. When $\mu > 0$, Faraday's law in vector differential form, $\vec{k} \times \vec{E} = \mu\omega\vec{H}$ denote a right-handed orthogonal triad whereas $\mu < 0$, $\{\vec{k}, \vec{E}, \vec{H}\}$ constitute a left-handed triad, which is explained by the vector differential mode exactly. We presented the capability of *Mathematica* for vector analysis and visualizing vector fields in previous works [19,20]. In this work, we have presented a platform visualizing vector fields added by the field densities using *Mathematica*. The platform will manipulate vi-

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visualizing of five vector fields: $\vec{E} = -\vec{\nabla}\phi$, $\vec{D} = \epsilon\vec{E}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{H} = -\vec{\nabla}\phi^*$, and $\vec{B} = \mu\vec{H}$ in matter. It is composite of several modules to visualize the electromagnetic vector fields and specify the electromagnetic characteristics of the comparable modes of the material. Therefore, it is potentially useful as a pedagogical tool for the electromagnetism in the higher class of undergraduate. It is assumed that the reader has some familiarity with vector differential operators in electrostatics. The basic background of electrostatics (EM) is reviewed briefly in Section II. In Section III, the plots of vector fields are produced in *Mathematica* and demonstrated using graphics. In Section IV, a summary is provided based on the platform of vector fields. The reader who is conversant with vector differential mode in EM may skip Section II. In Appendix, few segments of *Mathematica* 11.0 code are presented to aid the reader's comprehension of vector fields plotting.

II. Vector differential operators for vector fields in electrostatics

1. Electric vector field from scalar potential

(1) Electric field in vacuum

Electrostatics is the study of the time-independent distributions of electric charges and electromagnetic fields in electrodynamics [1-6]. If an arbitrary charge q is placed in an electric field $\vec{E}$, it will experience a force $\vec{F}$ given by $\vec{F} = q\vec{E}$. The electric field exerted to test charge q can also be expressed as an integral over the charge density instead of the total charge: $Q = \int \rho(\vec{r}') dv'$

$$\begin{aligned} \vec{F}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{qQ}{|\vec{r}-\vec{r}'|^2} \hat{e}_r = q \underbrace{\frac{1}{4\pi\epsilon_0} \int_V \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \rho(\vec{r}') dv'}_{\vec{E}(\vec{r})} \\ &= -q \vec{\nabla} \left(\underbrace{\frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\vec{r}-\vec{r}'|} \rho(\vec{r}') dv'}_{\phi(\vec{r})} \right) \\ \blacktriangleright \vec{E} &= -\vec{\nabla}\phi(\vec{r}), \text{ with } \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dv', \quad (1) \end{aligned}$$

where dv' is used to designate an element of volume in the charge distribution and V denote the entire volume occupied by the charge distribution. We get Gauss's law in vector differential operator from Eq. (1) by the divergence theorem;

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \text{ or } \vec{\nabla}\phi \cdot \hat{n} = -\frac{\sigma}{\epsilon_0} \quad (2)$$

As since $\vec{E} = -\vec{\nabla}\phi$, Gauss's law guide to Poisson equation $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$. In regions of no charge, Poisson's equation reduce to Laplace equation: $\nabla^2\phi = 0$. The general solution of Laplace equation in spherical coordinates for a problem with azimuthal symmetry is as follows [5,6]:

$$\blacktriangleright \phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta), \quad (3)$$

where $P_l(\cos\theta)$ represents the Legendre polynomials. Coefficients A_l and B_l can be determined from the boundary conditions.

(2) Constitutive relation in dielectrics: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

For the potential of an arbitrary localized charge distribution $d\rho(\vec{r}')$ in power of $\frac{1}{r}$, we can write potential at point $\vec{r}$ [1-3,5] as :

$$\begin{aligned} \phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int \left(\frac{1}{r} \right) \rho dv' \\ &= \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{r} \int \rho dv' + \frac{1}{r^2} \int r' \cos\theta \rho dv' \right. \\ &\quad \left. + \frac{1}{r^3} \int (r')^2 \left(\frac{2}{3} \cos^2\theta - \frac{1}{2} \right) \rho dv' + \dots \right] \quad (4) \end{aligned}$$

Here we use the binomial expansion of Legendre polynomials: $\frac{1}{r} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos\theta)$. The first term is called the monopole term, 2nd the dipole term, and 3rd quadrupole term. In this case we take the dipole term as a continuous charge distribution. The potential due to the dipole moment can be written succinctly as:

$$\phi_p(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \cdot \int \vec{r}' \rho dv' = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3}, \quad (5)$$

where $\vec{p} = \int \vec{r}' \rho dv'$ is called the dipole moment of the distribution. A convenient measure of this effect is the polarization $\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_i \vec{p}_i$, then $\vec{p} = \vec{P} dv$ [1,2].

And apply the vector identity $\vec{\nabla} \cdot \vec{P} \left(\frac{1}{r} \right) = \frac{1}{r} \vec{\nabla} \cdot \vec{p} +$

$\vec{p} \cdot \vec{\nabla} \left(\frac{1}{r} \right)$. Then the potential in the dielectrics with $\vec{P}$ instead of $\vec{p}$ in Eq. (5):

$$\begin{aligned}\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_v \frac{\vec{P} \cdot \vec{r}}{r^3} dv = \frac{1}{4\pi\epsilon_0} \int_v \vec{P} \cdot \vec{\nabla} \left(\frac{1}{r} \right) dv \\ &= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r} \int_s \vec{P} \cdot \hat{n} da - \int_v \frac{1}{r} (\vec{\nabla} \cdot \vec{P}) dv \right] \\ &= \frac{1}{4\pi\epsilon_0} \int_s \frac{1}{r} \sigma_b da + \frac{1}{4\pi\epsilon_0} \int_v \frac{1}{r} \rho_b dv,\end{aligned}\quad (6)$$

where $\sigma_b \vec{P} \cdot \hat{n}$ is the bounded surface charge and $\rho_b = -\vec{\nabla} \cdot \vec{P}$ is the bounded volume charge due to the polarization of the dielectric media. When applying Gauss's law to a region containing the charges embedded in a dielectric, the polarization charges (ρ_b) as well as the charge (ρ) must be included. Gauss's law in dielectric medium is

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} (\rho + \rho_b) = \frac{1}{\epsilon_0} (\rho - \vec{\nabla} \cdot \vec{P}) \\ \Rightarrow \quad \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) &= \vec{\nabla} \cdot \vec{D} = \rho,\end{aligned}\quad (7)$$

where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is a new macroscopic field vector, the electric displacements $\vec{D}$, that act as the constitutive relation in dielectrics. $\vec{D}$ having the same unit as $\vec{P}$ charge per unit area (C/m²), for $\oint \vec{D} \cdot \hat{n} da = Q$.

2. Magnetic vector field from the potentials

(1) Magnetic vector field: $\vec{B} = \vec{\nabla} \times \vec{A}$

Using the Biot-Savart law, the force $\vec{F}_2$, exerted on circuit 2 of current I_2 owing to the influence of circuit 1 in current I_1 is as follows:

$$\begin{aligned}\vec{F}_2 &= \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)]}{|\vec{r}_2 - \vec{r}_1|^3} \\ &= I_2 \oint_2 d\vec{l}_2 \times \frac{\mu_0}{4\pi} I_1 \oint_1 \frac{d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \\ &= I_2 \oint_2 d\vec{l}_2 \times \vec{B}(\vec{r}_2)\end{aligned}\quad (8)$$

Next, using the vector identity $\vec{\nabla} \times (\phi \vec{F}) = \vec{\nabla} \phi \times \vec{F} + \phi \vec{\nabla} \times \vec{F}$, we can write as follows:

$$\vec{B}(\vec{r}_2) = \frac{\mu_0}{4\pi} \int_{v_1} \vec{J}(\vec{r}_1) \times \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} dv_1$$

$$\begin{aligned}&= -\frac{\mu_0}{4\pi} \int_{v_1} \vec{J}(\vec{r}_1) \times \nabla_2 \left(\frac{1}{|\vec{r}_2 - \vec{r}_1|} \right) dv_1 \\ &= \vec{\nabla}_2 \times \frac{\mu_0}{4\pi} \int_{v_1} \frac{\vec{J}(\vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} dv_1 = \vec{\nabla}_2 \times \vec{A}(\vec{r}_2)\end{aligned}\quad (9)$$

where the calculation $\vec{\nabla}_2 \vec{J}(\vec{r}_1) = 0$. It confirms that $\vec{B}(\vec{r}_2) = \vec{\nabla}_2 \times \vec{A}(\vec{r}_2)$ with $\vec{A}(\vec{r}_2)$

$$\blacktriangleright \vec{A}(\vec{r}_2) = \frac{\mu_0}{4\pi} \int_{v_1} \frac{\vec{J}(\vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} dv_1.\quad (10)$$

The vector potential $\vec{A}(\vec{r}_2)$ may be applied to current circuits by making the substitution $\vec{J} dv = \vec{I} d\vec{r}$ and use an expand in powers of r_1/r_2 to get $|\vec{r}_2 - \vec{r}_1|^{-1} = \frac{1}{r_2} \left(1 + \frac{\vec{r}_1 \cdot \vec{r}_2}{r_2^2} + \dots \right)$ and vector identity $d\vec{r}_1 (\vec{r}_1 \cdot \vec{r}_2) = \frac{1}{2} (\vec{r}_1 \times d\vec{r}_1) \times \vec{r}_2$ [1]. Then Eq. (10) is

$$\begin{aligned}\vec{A}(\vec{r}_2) &= \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \\ &= \frac{\mu_0 I}{4\pi} \frac{1}{r_2} \oint d\vec{r}_1 + \frac{1}{r_2^3} \oint d\vec{r}_1 (\vec{r}_1 \cdot \vec{r}_2) + \dots \\ &\simeq \frac{\mu_0}{4\pi} \left[\frac{1}{2} \oint (\vec{r}_1 \times d\vec{r}_1) \times \frac{\vec{r}_2}{r_2^3} \right] = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}_2}{r_2^3},\end{aligned}\quad (11)$$

where $\vec{m} = \frac{1}{2} \oint (\vec{r}_1 \times d\vec{r}_1)$ is the magnetic dipole moment of the circuit.

(2) Constitutive relation in magnetic media: $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

Now we define a macroscopic vector point function magnetization: $\vec{M} = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_i m_i$. Each volume element ΔV of magnetized matter is characterized by a magnetic dipole moment: $\vec{m} = \vec{M} dv'$ to explore the influence of matter on the magnetic field. We can write $\vec{B}$ with $\phi_M^*(\vec{r})$ and $\vec{M}$ from Eq. [11]:

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \int_{v_0} \vec{\nabla} \times \left(\frac{\vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) dv' \\ &= \frac{\mu_0}{4\pi} \int_{v_0} \left[\vec{M}(\vec{r}') \vec{\nabla} \cdot \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} - (\vec{M}(\vec{r}') \cdot \vec{\nabla}) \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] dv'\end{aligned}$$

$$= \mu_0 \vec{M}(\vec{r}) - \mu_0 \vec{\nabla} \left(\underbrace{\frac{1}{4\pi} \int_{V_0} \vec{M}(\vec{r}') \cdot \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dv'}_{\phi_M^*(\vec{r})} \right)$$

$$\vec{B}(\vec{r}) = \mu_0 \vec{M}(\vec{r}) - \mu_0 \vec{\nabla} \phi_M^*(\vec{r}) = \mu_0 (\vec{M}(\vec{r}) + \vec{H}(\vec{r})), \quad (12)$$

where magnetic scalar potential $\phi_{M(r)}^*$ is

$$\phi_{M(r)}^* = \frac{1}{4} \int_{V_0} \frac{\vec{M}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dv' \quad (13)$$

Here an auxiliary magnetic vector field, $\vec{H}$, is introduced as $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$, i.e., $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, this is the constitutive equation in magnetic medium calculated from the magnetic scalar potentials: $\vec{H}(\vec{r}) = -\vec{\nabla} \phi_M^*(\vec{r})$. Using the vector identity $\vec{\nabla}' \cdot \left(\frac{\vec{M}}{|\vec{r} - \vec{r}'|} \right) = \vec{\nabla}' \cdot \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \cdot \vec{M} + \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \cdot \vec{M}$, we can present $\phi_M^*(\vec{r})$ with the pole densities σ_M and ρ_M [1-3,5].

$$\begin{aligned} \phi_M^*(\vec{r}) &= \frac{1}{4\pi} \int_{V_0} \vec{M}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) dv' \\ &= \frac{1}{4\pi} \int_{V_0} \vec{\nabla}' \cdot \left(\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \cdot \vec{M}(\vec{r}') dv' \\ \phi_M^*(\vec{r}) &= \frac{1}{4\pi} \int_{S_0} \frac{\sigma_M(\vec{r}')}{|\vec{r} - \vec{r}'|} da' + \frac{1}{4\pi} \int_{V_0} \frac{\rho_M(\vec{r}')}{|\vec{r} - \vec{r}'|} dv', \quad (14) \end{aligned}$$

where $\sigma_M(\vec{r}') = \vec{M}(\vec{r}')$ is the surface density of magnetic pole strength and $\rho_M(\vec{r}') = -\vec{\nabla}' \cdot \vec{M}(\vec{r}')$ is the magnetic pole density obtained from an integral over the entire volume of material V_0 in Eq. (14), which contrast with the potential from the dipole moment $\phi(\vec{r})$ in Eq. (6).

(3) Maxwell equation in matter

The Maxwell's equations describing the vector fields in macroscopic media can be written in the vector differential form at the MKSA unit system [1-3,5] as :

$$\vec{\nabla} \cdot \vec{D} = \rho, \quad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (15)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (16)$$

where $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$. In dielectric media, polarization $\vec{P} = \epsilon_0 \chi \vec{E}$ is defined and that $\vec{D} = \epsilon_0 (1 + \chi) \vec{E} = \epsilon \vec{E}$, where χ is the electric susceptibility of a material. The dielectric constant K is written by $K = \frac{\epsilon}{\epsilon_0}$ and $\vec{D} = \epsilon \vec{E} = K \epsilon_0 \vec{E}$. In magnetic material, magnetization is described by $\vec{M} = \chi_m \vec{H}$, then $\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 K_m \vec{H} = \mu \vec{H}$ where $K_m = 1 + \chi_m = \frac{\mu}{\mu_0}$. K_m is the relative permeability. As $\vec{\nabla} \cdot \vec{B} = 0$, and $\nabla^2 \phi_M^* = 0$, if μ is at least piecewise constant in each region, the magnetic scalar potential ϕ_M^* also satisfies Laplace's equation [5,6]. To visualize vector fields, we must determine the potentials from the Eqs. (1), (6), (10), and (14) with zonal harmonics. Eq. (3) is applicable in spherical boundaries and azimuthal symmetry applications. Thus we can calculate and plot the vector fields appropriately on the medium environments.

III. Visualizing vector fields with *Mathematica*

1. Plotting $\vec{E} = -\vec{\nabla} \phi$

For the visualization of vector fields, *Mathematica* supports many procedures including `VectorFieldPlot`, `GradientFieldPlot`, and `ContourPlot`. `VectorFieldPlot[{fx, fy}]` plots a vector function $\{f_x, f_y\}$, and `GradientFieldPlot[(f)]` plots the gradient of scalar function f . `VectorFieldPlot[{x,y}]` and `GradientFieldPlot[(x2+y2)]` plots are shown in Fig. 1(a) column. They are simpler but instructive examples that show the same vector fields results from `PlotVectorField[{x,y}]` and `Plot GradientField[(x2+y2)]`. $\vec{E} = -\vec{\nabla} \phi$ is shown in bottom row of Fig. 1. We have used the userdefined procedure `heejyPlot` (in *Mathematica* code 3) to plot and manipulate the electric and magnetic fields with the prescribed scalar potential (`In[451-458]` in *Mathematica* code 2) appropriate to the material medium more effectively. The procedure includes the modules `GradientFieldPlot` and `ContourPlot`, and material parameters available to the prescribed potentials are assigned. When we assigned a potential the procedure produce a vector field platform as in Fig. 2. The plots of $-\vec{\nabla} \phi$ are produced with the `GradientFieldPlot` function in *Mathematica*. Complete modules are available in Ref. [24].

Figure 1. (Color online) Plots of the vector fields (top row) and gradient of scalar fields (bottom row). Plot (a) Vector{x,y} is generated using the function VectorFieldPlot[{x,y}] and $\nabla(x^2+y^2)$ (bottom row) is generated using GradientFieldPlot[{x^2+y^2}] in *Mathematica*. The other plots are generated using the same method used in (a).

Figure 2. (Color online) Plots of $\vec{E} = -\nabla\phi$ of the two monopole potentials generated by function GradientFieldPlot in *Mathematica*.

2. Plotting $\vec{B} = \nabla \times \vec{A}$

It is helpful to calculate vectors in both Cartesian and cylindrical coordinates in *Mathematica* as in *Mathematica* code 1. If circuit current is confined in wire the vector potential $\vec{A}_2(\vec{r})$ is determined by Eq. (10), then we can calculate magnetic induction $\vec{B}$ with curl of vector potential $\vec{B}(\vec{r}) = \nabla \times \vec{A}_2(\vec{r})$ and plot vector fields in both 2D and 3D graphics with VectorFieldPlot3D in *Mathematica* as shown Fig. 3. The calculated $\vec{B}$ field is $B_{X_{\infty}} = \left\{ -\frac{2I_0 y}{(x^2 + y^2)}, \frac{2I_0 x}{(x^2 + y^2)}, 0 \right\}$, in Cartesian coordinates while $B_{C_{\infty}} = \left[0, -\frac{2I_0}{r}, 0 \right]$ in cylind-

rical coordinates in *Mathematica* code 2. This indicates $\vec{B} = \nabla \times \vec{A}$ fields have x and y component in Cartesian while only θ component in cylindrical coordinates when $\vec{A}_z(\vec{r})$ have a z component, which is the characteristics of $\vec{B} = \nabla \times \vec{A}$. If, instead of being confined to wire, the current exists in a medium, then $\vec{A}_z(\vec{r}_2)$ is represented with the magnetic dipole moment $\vec{m} = \frac{1}{2} \oint (\vec{r}_1 \times d\vec{r}_2)$ as in Eq. (11).

3. Prescribed potentials for vector fields

(1) Polarized sphere with polarization $\vec{P}$ in vacuum

The electric fields generated by a uniformly polarized sphere of radius a with polarization $\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_i p_i$ or $\vec{p} = \vec{P} dv$. The potential due to $\vec{P}$ is given by Eq. (6),

$$\phi(\vec{r}, \theta) = \frac{1}{4\pi\epsilon_0} \int_s \frac{1}{r} \sigma_b da = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_s P \cos\theta da, \quad (17)$$

where $\sigma_b = \vec{P} \cdot \hat{n} = P \cos\theta$, and the volume bounded charge density $\rho_b = -\nabla \cdot \vec{P} = 0$ because $\vec{P}$ is the constant polarization vector. Laplace's equation in spherical coordinates is solved by zonal harmonics with spherical boundaries and azimuthal symmetry as follows [1,2];

$$\phi_1(r, \theta) = \sum_{l=0}^{\infty} [B_l r^{-(l+1)}] P_l(\cos\theta), \quad r \geq a \quad (18)$$

Figure 3. (Color online) Plots of vector potential $\vec{A}_z(\vec{r})$ and vector field $\vec{B} = \vec{\nabla} \times \vec{A}_z(\vec{r})$ in 2D and 3D. These vector fields were visualized using the function `VectorFieldPlot3D` in *Mathematica*.

$$\phi_2(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l P_l(\cos\theta)], \quad r \leq a \quad (19)$$

At $r=a$, the boundary conditions: $\phi_1(r, \theta) = \phi_2(r, \theta)$ give the relation $A_1 a^1 = B_1 \frac{1}{a^{1+1}}$, and $\vec{\nabla} \phi_{1r} - \vec{\nabla} \phi_{2r} = -\frac{\sigma(\theta)}{\epsilon_0}$ by Eq.(2). From these we can get $A_1 = \frac{P}{3\epsilon_0}$ by the “eyeball” method; $\sigma(\theta)_b = \vec{P} \cdot \hat{n} = P \cos[\theta]$. Thus, the potentials as follows:

$$\blacktriangleright \phi_{p1}(r, \theta) = \frac{P}{3\epsilon_0} \frac{a^3}{r^2} \cos[\theta], \quad r \geq a \quad (20)$$

$$\blacktriangleright \phi_{p2}(r, \theta) = \frac{P}{3\epsilon_0} r \cos[\theta], \quad r \leq a \quad (21)$$

Next, we obtain the electric fields inside and outside of the uniformly polarized sphere; $E_1 = -\vec{\nabla} \phi_1(r, \theta)$ $E_2 = -\vec{\nabla} \phi_2(r, \theta)$, and plot as Fig. 5(a).

(2) Grounded spherical conductor in electric field $\vec{E}_0$

As a canonical problem in physics, we considered a conducting sphere radius a held at constant (let zero) potential and placed in a uniform field $\vec{E}_0$ [1,5,6,10]. The potential can be written as an expansion with Legendre polynomials by Eq. (3). The coefficients $\{A_l, B_l\}$ were obtained from the boundary conditions. The electric field should be $\vec{E}_0$ as $r \rightarrow \infty$. So that the potential must be $\phi_1 = -\int \vec{E}_0 \cdot d\vec{l} = -E_0 r \cos[\theta]$. Then $\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos[\theta]$, therefore all $A_l = 0$ except $A_1 = -E_0$. At $r=a$ $\phi=0$,

$A_l a^l + B_l a^{-(l+1)} = 0 \Leftrightarrow B_1 = -A_1 a^3 = E_0 a^3$. Thus we determined the potential:

Figure 4. (Color online) Vector field intensity platform for `epvx[z,y]` using `huefunc` $\left[\frac{\text{Norm}[\text{epvx}[z,y]]}{\text{epvxmax}} \right]$ with `ColorFunctions` shows the results as `epvx[1.15a, 0.75a] = {2.12012, -2.07063}` and `ecvx[1.15a, 0.75a] = {4.3516, 0}`, respectively.

$$\blacktriangleright \phi_c(r, \theta) = -E_0 (r \cos[\theta] - \frac{a^3}{r^2} \cos[\theta]) \quad (22)$$

Next, we get $\vec{E}_1 = -\vec{\nabla} \phi(r, \theta)$ and plot as Fig. 5(b). The field intensity is determined using `huefunc` $\left[\frac{\text{Norm}[\text{epvx}[z,y]]}{\text{epvxmax}} \right]$ with `ColorFunctions`. The field intensity platform is manipulated with the materials (K values) and medium environment (initial fields) and calculations of the vector field intensity at the vector points ($r=0.75a, 1.15a$) are shown in Fig. 4.

(3) Dielectric sphere in an initially uniform electric field $\vec{E}_0$

The potential of the dielectric sphere with a radius of a in initially uniform electric field $\vec{E}_0$ may be expressed as a sum of zonal harmonics in Eq. (18) [1,2]. As $r \rightarrow \infty$, $\vec{E} \rightarrow \vec{E}_0$ and the potential on surface

(I). $\vec{E}$ & $\vec{D}$ Fields (II). Field density of $\vec{E}$ & $\vec{D}$

Figure 5. (Color online) Plots of vector fields (I) and the vector field densities (II) of $\vec{E}$ and $\vec{D}$ for a dielectric sphere of radius a . (I) The vector field $\vec{E}[z, y]$ of a polarized sphere with $P = 5.5 \times 10^{-11} \text{ C/m}^2$ and $\vec{E}[z, y]$ and $\vec{D}[z, y]$ for three dielectric constants, $K = \{\infty, 2.3, 2.3\}$, in an initially uniform electric field $\vec{E}_0 = 2.15 \text{ V/m}$. (II) The vector field densities of $\vec{E}[z, y]$ and $\vec{D}[z, y]$ under the same conditions as (I). The vector field densities are estimated from the maximum fields of $\{E_{pmax} = 1.0 \text{ V/m}, E_{cmax} = 3.15 \text{ V/m}, E_{dmax} = 3.15 \text{ V/m}, \text{ and } D_{dmax} = 3.15 / \text{Cm}^2\}$.

should be continuous. The potential is presented by two harmonics of the lowest-order for the outside and inside of dielectric sphere, respectively as follows:

$$\phi_1(r, \theta) = A_1 r \cos[\theta] + C_1 r^{-2} \cos[\theta], \quad r \geq a \quad (23)$$

$$\phi_2(r, \theta) = A_2 r \cos[\theta] + C_2 r^{-2} \cos[\theta], \quad r \geq a \quad (24)$$

The constants A_1 , A_2 , C_1 , and C_2 determined from the boundary conditions as $A_1 = -E_0$, $C_2 = 0$, $A_2 = -\frac{3E_0}{K+2}$, and $C_1 = \frac{K-1}{K+2} E_0 a^3$. Then we determine the electric scalar potentials:

$$\blacktriangleright \phi_{e1}(r, \theta) = E_0 \frac{(K-1)}{(2+K)} \frac{a^3}{r^2} \cos[\theta] - E_0 r \cos[\theta], \quad r \geq a \quad (25)$$

$$\blacktriangleright \phi_{e2}(r, \theta) = -\frac{3E_0}{2+K} r \cos[\theta], \quad r \geq a \quad (26)$$

For $\vec{D} = \epsilon \vec{E} = K\epsilon_0 \vec{E}$, we obtained the dielectric scalar potentials ϕ_{d1} and ϕ_{d2} as $\phi_{d1}(r, \theta) = \epsilon_0 \phi_{e1}$, $\phi_{d2}(r, \theta) = K\epsilon_0 \phi_{e2}$. We get $\vec{E}_1$ and $\vec{E}_2$ from the ϕ_{e1} and ϕ_{e2} in an initially uniform field of $\vec{E}_0$ as follows:

$$\vec{E}_1 = -\nabla \phi_{e1} = \left\{ \left(E_0 + \frac{K-1}{K+2} \frac{a^3}{r^3} \cos[\theta] \right), \right.$$

$$\left(-E_0 + \frac{K-1}{K+2} \frac{a^3}{r^3}\right) \sin[\theta], 0\right\}, \quad r \geq a \quad (27)$$

$$\vec{E}_2 = -\vec{\nabla} \phi_{e2} = \frac{3E_0}{K+2} \{ \cos[\theta], -\sin[\theta], 0 \}, \quad r \leq a \quad (28)$$

We can write $\vec{E}_2 = \frac{3}{K+2} E_0 \hat{e}_z$ in Cartesian coordinate, where $\hat{e}_z = \cos[\theta] \hat{e}_r - \sin[\theta] \hat{e}_\theta$. This can be rewritten as $\vec{E}_2 = \left(1 - \frac{K-1}{K+2}\right) E_0 \hat{e}_z$, which represent the depolarization (Clausius-Mossotti factor $\frac{K-1}{K+2}$) [1,2]. The molecular polarizability is $\alpha = \frac{3\epsilon_0}{N} \frac{K-1}{K+2}$, where N is number of atoms or molecules per volume. Thus, characteristics of the $\vec{E}_2$ inside the dielectric give us important quantitative information about the structure of the atoms [21]. The fields $\vec{E}$ and $\vec{D}$ inside and outside of the dielectric sphere on the K values are shown in Fig. 5.

(4) Uniformly magnetized sphere in a nonpermeable medium

If the current density vanishes in some finite region of space, $\vec{\nabla} \times \vec{H} = 0$ by Ampere's law (Eq. (16)). This implies that we can introduce a magnetic scalar potential ϕ_M such that $\nabla^2 \phi_M = 0$. As a common practical situation, we consider the hard ferromagnets with magnetized $\vec{M}$. For such material, fixed magnetization $\vec{M}(\vec{x})$. Then, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times (\vec{H} + \vec{M}) = 0$ becomes a magnetostatic Poisson's equation $\nabla^2 \phi_M = -4\pi \rho_M$ with a magnetic charge density ρ_M if there is no boundary surface. The magnetic scalar potential ϕ_M due to the magnetization $\vec{M}$ is given by [1,5]:

$$\phi_M(r, \theta) = \frac{1}{3} M a^2 \frac{r <}{r^2 >} \cos[\theta] \quad (29)$$

Thus, the magnetic potential is

$$\blacktriangleright \phi_{M1}(r, \theta) = \frac{1}{3} M \frac{a^3}{r^2} \cos[\theta], \quad r \geq a \quad (30)$$

$$\blacktriangleright \phi_{M2}(r, \theta) = \frac{1}{3} M r \cos[\theta], \quad r \leq a \quad (31)$$

$\vec{M} = M \hat{e}_z$ is parallel to the z-axis. For the magnetic induction in the magnetized sphere, we may estimate

a modified the magnetized scalar potential ϕ_{M2}^* in the sphere satisfying $\vec{B}_2^* = -\vec{\nabla} \phi_{M2}^* = \mu_0 (\vec{H} + \vec{M}) = \mu_0 \frac{2}{3} \vec{M}$,

thus, we can determine ϕ_{M2}^* as follows:

$$\blacktriangleright \phi_{M2}^*(r, \theta) = -\mu_0 \frac{2}{3} M r \cos[\theta], \quad r \leq a \quad (32)$$

Then $\vec{H}_2 = -\vec{\nabla} \phi_{M2} = -\frac{1}{3} \vec{M}$ and $\vec{B}_2^* = -\vec{\nabla} \phi_{M2}^* = \mu_0 \frac{2}{3} \vec{M}$. Here, we note that $\vec{B}_2^*$ is parallel to $\vec{M}$ while $\vec{H}_2$ is antiparallel to $\vec{M}$ as shown in Fig. 6(a).

(5) Uniformly magnetized sphere in uniform magnetic field $\vec{B}_0$

Consider a sphere of linear magnetic material with a radius a and a permeability μ in space containing an initially uniform magnetic field $\vec{B}_0$. The boundary conditions are applied to obtain the parameters of the Eq. (23): (i) as $r \rightarrow \infty$, $\vec{B}_1 = \vec{B}_0$ from $-\mu_0 \vec{\nabla} \phi_2 = \vec{B}_0$,

we get $A_1 = \frac{-B_0}{\mu_0}$. (ii) as $r \rightarrow 0$, $C_2 \rightarrow 0$ and (iii)

$\vec{B}_{1r} = \vec{B}_{2r}$ at $r = a$, from those $\left\{ C_1 \rightarrow \frac{a^3 B_0 (-1 + Km)}{(2 + Km) \mu_0}, A_2 \rightarrow -\frac{3B_0}{(2 + Km) \mu_0} \right\}$ [1,5]. Thus, we get the magnetic potentials:

$$\blacktriangleright \phi_{m1}(r, \theta) = \frac{B_0}{\mu_0} \left[\frac{(-1 + Km) a^3}{2 + Km} \frac{1}{r^2} - r \right] \cos[\theta], \quad r \geq a \quad (33)$$

$$\blacktriangleright \phi_{m2}(r, \theta) = -\frac{B_0}{\mu_0} \frac{3}{(2 + Km)} r \cos[\theta], \quad r \leq a \quad (34)$$

Then, we get $\vec{B}_1 = \mu_0 \vec{H}_1 = \mu_0 (-\vec{\nabla} \phi_{m1})$ and $B_2 = \mu H_2 = \mu (-\vec{\nabla} \phi_{m2})$ in the spherical system as follow:

$$\vec{B}_1 = B_0 \left\{ \frac{(-1 + Km) 2a^3 \cos[\theta]}{(2 + Km) r^3} + \cos[\theta], \frac{(-1 + Km) a^3 \sin[\theta]}{(2 + Km) r^3} - \sin[\theta], 0 \right\}, \quad r \leq a \quad (35)$$

$$\vec{B}_2 = Km \left\{ \frac{3B_0 \cos[\theta]}{2 + Km}, -\frac{3B_0 \sin[\theta]}{2 + Km}, 0 \right\}, \quad r \geq a \quad (36)$$

where $\vec{B}_2 = Km \frac{3B_0}{2 + Km} \hat{e}_z$ parallel to the pole. Vector

(I). H & B Fields (II). $\vec{H}$ & $\vec{B}$ Fields Density

Figure 6. (Color online) Plots of the vector fields (I) and the vector field densities (II) of $\vec{H}$ and $\vec{B}$ for a magnetized sphere of radius a . (I) The vector field $\vec{H}[z, y]$ for a magnetized sphere of $M0 = 0.97$ T, the vector field $\vec{B}[z, y]$ for a magnetized sphere of $M0 = 77,190$ T in vacuum, and the vector fields, $\vec{H}[z, y]$ and $\vec{B}[z, y]$ for a magnetic material with $Km = 2.2$ in an initially uniform vector field $\vec{B}_0$. (II) The vector field densities of $\vec{H}[z, y]$ and $\vec{B}[z, y]$ under the same conditions as in (I). The vector field densities are estimated from the maximum fields $H_{max} = 4\pi \times 10^5$ A/m and $B_{max} = \pi$ T, respectively, to compare the fields in a platform. $M0 = \mu_0 M$, where M is the saturation magnetization of the ferromagnet [1].

fields of $\vec{B}_1$ and $\vec{B}_2$ are plotted on the Km values in the Cartesian coordinate in Fig. 6. Thus, we can calculate $\vec{B}_1$ and $\vec{B}_2$ with magnetic scalar potentials conveniently rather than with vector potentials.

(6) Prescribed magnetic scalar potentials

We can visualize the electromagnetic vector fields with heejyPlot user defined procedure composed of GradientFieldPlot and ContourPlot using the prescribed scalar potentials (► marked). We then summarize the electromagnetic scalar potentials and vector field functions in *Mathematica* code 1 that are utilized in the heejyPlot function. We can use the above pre-

scribed eight potentials (In[451-458]) of Cartesian coordinates to plot the vector fields in the heejyPlot procedure. By changing the parameters (K , Km , M , P) of the medium, plot the vector fields can be plotted repeatedly using the prescribed potentials shown as in Figs. 5 and 6.

4. Vector field plotting with scalar potentials

(1) Electric field plotting in the matter

Visualization of the electric fields in matter is presented two material medium: (i) the electric fields of the near the material in vacuum and (ii) in a uniform electric field $\vec{E}_0$. In vacuum, consider of the

electric field produced by a uniformly polarized sphere of radius a and polarization $\vec{P}$. To plot the electric fields ($\vec{E}_1$, $\vec{E}_2$) inside and outside the uniformly polarized sphere in vacuum where no other electric fields are present, we employed the prescribed potential `potp12`[y_- , z_-]. For plotting the electric fields ($\vec{E}_1$, $\vec{E}_2$) and displacements ($\vec{D}_1$, $\vec{D}_2$) fields inside and outside the material sphere we have assigned the prescribed potentials: `potc1` [y_- , z_-], `pote12` [y_- , z_-], `potd56` [y_- , z_-] to the `heejyPlot` procedure. The prescribed potentials are described in the *Mathematica* code 1. According to the selection of potential and parameter procedure plot the vector fields array as in Fig. 5. The adjustment of table index steps and plotting of additional vector field frames are possible by the index of table as in *Mathematica* code 2. It is evident that $\vec{E}_2 = -\frac{P}{3\epsilon_0}\hat{e}_z$ opposite to $\vec{P}$ and so $\vec{E}_2$ called the depolarization field [22,23]. Given that $\vec{D}_2 = \epsilon_0\vec{E}_2 + \vec{P}$ while $\vec{E}_2 = -\vec{\nabla}\phi_{e2}$ and $\vec{P} = \epsilon_0\chi\vec{E}_2$, we can examine $\vec{D}$ and $\vec{E}$ outside and inside sphere for the K values shown as Fig. 5. Visualization of the electric and the magnetic vector fields is presented using both vector field plotting and vector field density plotting with vector point field intensity values (see Field intensity at (II) in Figs. 5 and 6). The vector field densities are plotted using $\text{huefunc} = \frac{\text{Norm}[vfunc]}{vmax}$ with *ColorFunction* as in *Mathematica* code 2. These plot effectively reveal the features of the vector fields in matter. We can confirm the features of the vector fields by calculating the vector points inside and outside the sphere in the same platform. The intensity $\{E_1 > E_2, E_1 > E_2, E_1 > E_2, D_1 < D_2\}$ is shown in Fig. 5 and the array $\{H_1 > H_2, B_1 < B_1, H_2 > E_2, B_1 < B_2\}$ is shown in Fig. 6.

(2) Magnetic field plotting in the matter

For the plots of magnetic intensity ($\vec{H}_1$, $\vec{H}_2$) and magnetic induction ($\vec{B}_1$, $\vec{B}_2$) fields in vacuum, we employed the prescribed potentials: `poth42` [y_- , z_-], `potb43` [y_- , z_-] described in *Mathematica* code 1. To plot the magnetic intensity ($\vec{H}_1$, $\vec{H}_2$) and magnetic induction ($\vec{B}_1$, $\vec{B}_2$) in the uniform magnetic field $\vec{B}_0$, the prescribed potentials: `potm77` [y_- , z_-], `potm78` [y_- , z_-] are utilized `heejyPlot`. The plot frames can be expanded by controlling the table index of the platform similar to the

plotting of electric fields. In Fig. 6(a) we see that $\vec{H}_2 = -\frac{M}{3}\hat{e}_z$ is opposite to magnetization vector $\vec{M} = M\hat{e}_z$, that is the demagnetizing field, a result that in accord with $\vec{E}_2 = -\frac{P}{3\epsilon_0}\hat{e}_z$ in Fig. 5(a). All the visualization of the vector fields was done by plotting the vector fields instead of the illustrations described in most of texts [1-5]. It confirms that the constitutive relations: $\vec{B} = \mu_0(\vec{H} + \vec{M})$ and $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ are graphically represented in Figs. 5 and 6.

(3) Constitutive vector fields platform

Plotting the vector fields in different medium environments, we determined the eight functions for the scalar field potentials as presented in *Mathematica* code 1. Array of the vector fields is produced by selecting the prescribed potential and proper parameters, K , Km , P , M , E_0 , B_0 etc.. Next, an array of vector fields is automatically produced based on the suggested indexing. In this procedure eight arrays of vector fields are possible. Therefore, it is possible to test and examine the constitutive relations in both vector calculation and graphics mode simultaneously in the same platform. The difference between them could not be shown until vector field plots are produced in different medium environments. We have summarized the vector fields by visual representation of six fundamental vector fields as shown in Fig. 7. The platform presents the plotting of five kinds of vector fields and vector field density: $\vec{E} = -\vec{\nabla}\phi$, $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{H} = -\vec{\nabla}\phi^*$, $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, and $\vec{B} = \mu_0(\vec{H} + \vec{M})$, those are most fundamental vector fields in electromagnetics. Visualizing vector fields in *Mathematica* demonstrate the enhanced effectiveness of vector plots (eps, pdf) are scaled up in the graphics. Recently, artificial media i.e., metamaterials exhibit qualitatively new electromagnetic response functions which can not be found in the nature. Metamaterials are characterized by simultaneously negative permittivity and permeability, it referred to left-handed materials (LHMs). Their physical characteristics are also described with Maxwell equation Eq. (16) in vector differential mode [17,18]. Some aspects of *Mathematica* codes plotting vector fields in figures are presented in Appendix. Complete *Mathematica* codes `plotemvfs.nb` plotting the plots in the paper allows users to manipulate vector plots that are available from the site in Ref. [24]. Even if *Mathematica* is not installed, the user

Figure 7. (Color online) Vector field platform visualization of five vector fields from the vector potentials and scalar potentials: (a) $\vec{E} = -\vec{\nabla}\phi$, (b) $\vec{B} = \vec{\nabla} \times \vec{A}$, (c) $\vec{D}_2 = \epsilon_0 \vec{E}_2 + \vec{P}$ in an initially uniform electric field $\vec{E}_0$, (d) $\vec{H} = -\vec{\nabla}\phi^*$ and $\vec{B}_2 = \mu_0(\vec{H}_2 + \vec{M})$ side by side by $\vec{B}_2^*$, and norms (d-norms) corresponding to (d) fields. The sphere of $\vec{M}$ in $\vec{B}_2$ is embedded in a nonpermeable medium while the sphere of $\vec{M}$ in an initially uniform magnetic field $\vec{B}_0$. All the plots are selected from the Figs. 1-7. While one scale up the graphics on the platform examines the vector graphics more detail. The number to the left side is the time (seconds) to plot the 15 fields in *Mathematica* (CPU (dual-core):i3-6100@3.7GHz).

may use a plotemvfs.cdf file instead in CDF Player (free download) [25].

IV. Conclusion

We have visually represented the most fundamental

electromagnetic vector fields with the scalar and vector potentials using plotemvfs in *Mathematica*. In this work, eight user-defined functions for the prescribed potentials, were assigned, that were directly applied to the procedure heejyPlot to simultaneously plot the vector fields. This enables the examining and testing of the

vector fields in different media environment and confirm the constitutive relations: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ and $\vec{B} = \mu_0(\vec{H} + \vec{M})$ graphically. In this procedure, we manipulated only the vector field plotting of $\vec{E}$ and $\vec{B}$ separately unless they interacted with each other. Thus, we only considered the time independent terms in the Maxwell's equations Eq. (15). The visualization of electromagnetic vector fields was facilitated using the Poynting or Jones vectors in other studies [26,27]. Herein, we focused on visualizing the fundamental electric and magnetic vector fields separately and manipulate the plot of the vector fields in cross-checking with vector differential operators. According to our results, the plotemvfs platform could be a useful foundation for physics students and researchers to conceptualize, visualize, and quantize the law of physics knowledge in the knowledge and information converging process.

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APPENDIX A: Mathematica codes

The *Mathematica* codes used for the plots shown in Figs. 4-6 to interpret the producing the vector fields. However, with these codes the vector fields could not be produced. The vector fields are produced by the complete *plotemvfs* procedure (Ref. [24]).

Mathematica code 1

```
In[431]:= eq4[y_,z_]:=phic1,  $\phi_{c1}$  (Eq. (22))
In[432]:= eq31[y_,z_]:=phip1,  $\phi_{p1}$  (Eq. (20))
In[433]:= eq32[y_,z_]:=phip2,  $\phi_{p2}$  (Eq. (21))
In[434]:= eq1[y_,z_]:=phie1,  $\phi_{e1}$  (Eq. (25))
In[435]:= eq2[y_,z_]:=phie2,  $\phi_{e2}$  (Eq. (26))
In[436]:= eq41[y_,z_]:=phiM1,  $\phi_{M1}$  (Eq. (30))
In[437]:= eq42[y_,z_]:=phiM2,  $\phi_{M2}$  (Eq. (31))
In[438]:= eq43[y_,z_]:=phiM2s,  $\phi_{M2s}$  (Eq. (32))
In[439]:= eq5[y_,z_]:=phid1,  $\phi_{d1}$  (Eq. (25))
In[440]:= eq6[y_,z_]:=phid2,  $\phi_{d2}$  (Eq. (26))
In[441]:= eq70[y_,z_]:=phim1,  $\phi_{m1}$  (Eq. (33))
In[442]:= eq80[y_,z_]:=phim2,  $\phi_{m2}$  (Eq. (34))
In[443]:= eq7[y_,z_]:=mu0*phim1,  $\phi_{m1}$  (Eq. (33))
In[444]:= eq8[y_,z_]:=mu*phim2,  $\phi_{m2}$  (Eq. (34))
In[451]:= potc1[y_,z_]:=If[y^2+z^2>a,eq4[y,z],0];
```

```
In[452]:= potp12[y_,z_]:=If[y^2+z^2>a,eq32[y,z],eq31[y,z]];
In[453]:= pote12[y_,z_]:=If[y^2+z^2>a,eq2[y,z],eq1[y,z]];
In[454]:= poth42[y_,z_]:=If[y^2+z^2>a,eq42[y,z],eq41[y,z]];
In[455]:= potb43[y_,z_]:=If[y^2+z^2>a,eq43[y,z],eq41[y,z]];
In[456]:= potd56[y_,z_]:=If[y^2+z^2>a,eq6[y,z],eq5[y,z]];
In[457]:= potm77[y_,z_]:=If[y^2+z^2>a,eq80[y,z],eq70[y,z]];
In[458]:= potm78[y_,z_]:=If[y^2+z^2>a,eq8[y,z],eq7[y,z]];

```

Mathematica code 2

```
In[501]:= heejyPlot[potential,xlim,ylim,opts_]
:= Module[{plot1,plot2},
plot1= GradientFieldPlot[potential
,xlim,ylim,PlotPoints->18
,DisplayFunction->Identity];
plot2= ContourPlot[potential
,xlim,ylim,ContourShading->False
,PlotPoints->250]]
Show[plot1,plot2,opts];
heejyPlot[potential,opts_]:=
heejyPlot[potential,
,{x,1.1,-1.1},{y,1.1,-1.1} opts];
In[502]:= Table[
kd={1.0,2.3,3.5,4.3,5.0,6.1,28.4,80.1};
kemat={"K=1.00 Air","K=2.3 Polyethylene",
"K=3.5Nylon","K=4.3 Quartz","K=5.0 Glass",
"K=6.1NaCl","K=28.4Alcohol","K=80.1Water"};
In[503]:= Ec1=Show[heejyPlot[potc1[y,z]
,{z,-2.2,2.2},{y,-2.2,2.2},text1,text62
,DisplayFunction->Identity],circle1
,DisplayFunction->$DisplayFunction
,ImageSize->161];
Ep2=Show[heejyPlot[potp12[y,z],ibid.]]
Ein=Show[heejyPlot[pote12[y,z],ibid.]]
Din=Show[heejyPlot[potd56[y,z],ibid.]]
In[510]:= Row[{Ep2,Ec1,Ein,Din}
,{i,2,5,5}]/Framed
In[520]:= Row[{Ep2,Ec1,Ein,Din}
,{i,1,8,6}]/Framed
In[523]:= epv1=Grad[phip1{r,theta,phi},"Spherical"]
/.{r->1.15,P->5.5 10^-11} //Evaluate
In[524]:= epv2=Grad[phip2{r,theta,phi},"Spherical"]
/.{r->0.75,P->5.5 10^-11} //Evaluate
Out[510]= {-Graphics-} (Fig. 4)
Out[520]= {-Graphics-} (Fig. 5)
Out[523]= {2.2102,0,0}(epv1)
Out[524]= {-2.07063,0,0}(epv2)
```

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